On the Nakagami-*m* Crosscorrelation Function

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***Abstract*— The generic-order crosscorrelation function between two non-identically distributed Nakagami-*m* fading processes is derived, assuming a space-frequency diversity scenario with horizontally spaced omnidirectional antennas at the mobile station. Our result allows for distinct fading parameters, power imbalance, antenna spacing/angle, and frequency separation. Based on it, the coherence bandwidth and the coherence time/distance of the Nakagami-*m* channel are also obtained. The latter is found to be practically identical to the coherence time/distance of the Rayleigh channel.**

***Index Terms*— Crosscorrelation function, Nakagami-*m* channels.**

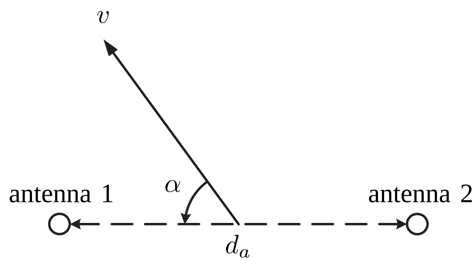

Fig. 1. Antenna configuration.

I. Introduction

Despite the many attractive aspects of the Nakagami-*m* distribution in describing the rapid fading statistics of the wireless channel, no unanimous dynamic model for the Nakagami-*m* channel has been established yet, either by physical argument or by empirical evidence. This may be partly attributed to the fact that such a model was not specified when the Nakagami-*m* distribution was originally proposed in [1]. In that work, only the (static) Nakagami-*m* random variable (RV) has been addressed, but not the (dynamic) Nakagami-*m* random process (RP). For this reason, most Nakagami-*m* simulators in the literature lack a physically-based dynamic model and have to make particular assumptions to accomplish the temporal correlation of the channel [2]–[6]. Besides, analytical expressions to the higher-order statistics resulting from these assumptions are usually unknown [2]. The higher-order statistics of the Nakagami-*m* process are indeed longstanding uncertainties and explicit discussion of this subject is rarely found in the literature.

A well-accepted physical model to the Nakagami-*m* process with fading parameter m multiple of half-integer relies on the decomposition of the squared fading envelope into the sum of $2m$ squared zero-mean independent identically distributed (IID) Gaussian fading processes [2], [7], [8]. In this paper, based on the above physical model, the generic-order crosscorrelation function (CCF) between two non-identically distributed Nakagami-*m* fading envelopes is derived, assuming a space-frequency diversity scenario with horizontally spaced omnidirectional antennas at the mobile station. Our result allows for distinct fading parameters, power imbalance, antenna spacing/angle, and frequency separation. Based on it, the coherence bandwidth and the coherence time/distance of the Nakagami-*m* channel are also obtained. The latter is found to be practically identical to the coherence time/distance of the Rayleigh channel. To the best of the authors' knowledge, no published analytical results on the Nakagami-*m* crosscorrelation exist yet. Original contributions of this work include the following:

1) Closed-form expression for the generic-order Nakagami-*m* CCF in a space-frequency diversity scenario;
2) Straightforward approach to account for space diversity, based on the time-space duality of fading phenomena;
3) Closed-form expression for the Nakagami-*m* crosscorrelation coefficient (CCC);
4) Simple, insightful, accurate approximation to the Nakagami-*m* CCC;
5) Evaluation of the coherence bandwidth of the Nakagami-*m* channel;
6) Evaluation of the coherence time/distance of the Nakagami-*m* channel.

II. The Nakagami-*m* Crosscorrelation Function

Consider two horizontally spaced omnidirectional antennas at the mobile station, as sketched in Fig. 1. The antenna spacing is d_a and the angle between the antenna axis and the vehicle speed v is $0 \le \alpha \le \pi/2$. We assume that the fading envelope $R_i(t)$, $i = 1, 2$, at the ith antenna is propagated at frequency ω_i (rad/s) and follows a Nakagami-*m* distribution with fading parameter m_i and mean power Ω_i. (Without loss of generality, let $m_1 \leq m_2$.) Next, we shall derive the generic-order crosscorrelation function $R_{R_1,R_2}(\tau) \triangleq E\left[R_1^k(t)R_2^l(t+\tau)\right]$ between $R_1(t)$ and $R_2(t)$ as

$$R_{R_1,R_2}(\tau) = \left(\frac{\Omega_1}{m_1}\right)^{k/2}\left(\frac{\Omega_2}{m_2}\right)^{l/2} \times \frac{\Gamma\left(m_1+k/2\right)\Gamma\left(m_2+l/2\right)}{\Gamma(m_1)\Gamma(m_2)} \times {}_2F_1\left(-\frac{k}{2},-\frac{l}{2};m_2;\rho_2(\tau)\right) \quad (1)$$

where

$$\rho_2(\tau) = \frac{J_0^2\left(2\pi\sqrt{(f_D\tau)^2+(d_a/\lambda)^2-2(f_D\tau)(d_a/\lambda)\cos\alpha}\right)}{1+(\Delta\omega\bar{T})^2} \quad (2)$$

${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ is the hypergeometric function, $f_D = v/\lambda$ is the maximum Doppler frequency in Hz for a vehicle speed v and carrier wavelength λ, $\Delta\omega = \omega_2 - \omega_1$ is the frequency separation in rad/s, and $\bar{T}$ is the mean time delay of the scattered waves. ($E[\cdot]$ denotes expectation.) For $m_1 = m_2$, $\Omega_1 = \Omega_2$, and $\Delta\omega = d_a = 0$, (1) specializes to the Nakagami-*m* autocorrelation function [9, Eq. (25)].

III. The Derivation Process

It is known that the sum of $2m$ squared zero-mean IID Gaussian RVs is a squared Nakagami-*m* RV with fading parameter m [1], [2], [7], [8]. Equivalently, a sum of m squared IID Rayleigh RVs can be utilized. In [1], this approach has been used to construct a bivariate model for Nakagami-*m* RVs having identical fading parameters, with their Rayleigh components assumed correlated two by two. The corresponding joint moments have been also obtained in that paper. More recently, [8] generalized these results for two non-identical Nakagami-*m* RVs R_i having parameters m_i and Ω_i, $i = 1, 2$. The generic-order joint moment of R_1 and R_2 has been derived in [8] as

$$E[R_1^k R_2^l] = \left(\frac{\Omega_1}{m_1}\right)^{k/2}\left(\frac{\Omega_2}{m_2}\right)^{l/2} \times \frac{\Gamma\left(m_1+k/2\right)\Gamma\left(m_2+l/2\right)}{\Gamma(m_1)\Gamma(m_2)} \times {}_2F_1\left(-\frac{k}{2},-\frac{l}{2};m_2;\rho_2\right) \quad (3)$$

where $m_1 \leq m_2$ and ρ_2 is the correlation coefficient between each correlated pair of squared Rayleigh RVs.

Although the above models concern RVs, there are no constraints for applying them to RPs. In other words, a squared Nakagami-*m* fading process with fading parameter m can be understood as a summation of m squared IID Rayleigh fading processes [7]. In the same way, a joint model for two correlated Nakagami-*m* processes is attained if the Rayleigh component processes are assumed correlated in groups of two. Correspondingly, the CCF of two Nakagami-*m* processes can be now obtained directly from (3) with the appropriate use of a physically-based CCC $\rho_2 = \rho_2(\tau)$ between two squared Rayleigh fading processes.

The statistical properties of Rayleigh fading processes have been extensively reported in the literature. In particular, the well-established formulation presented by Jakes [10] can be used. Using [10, Eqs. (1.5-11), (1.5-14), and (1.5-15)], it follows that

$$\rho_2(\tau) = \left(\frac{\mathrm{E}\left[D\left(\theta\right)\cos\left(\omega_D\tau\cos\theta-\Delta\omega T\right)\right]}{\mathrm{E}\left[D\left(\theta\right)\right]}\right)^2 + \left(\frac{\mathrm{E}\left[D\left(\theta\right)\sin\left(\omega_D\tau\cos\theta-\Delta\omega T\right)\right]}{\mathrm{E}\left[D\left(\theta\right)\right]}\right)^2 \quad (4)$$

where θ and T are the arrival angle and arrival time of the scattered waves, respectively, $D(\theta)$ is the horizontal directivity pattern of the receiving antenna, and $\omega_D = 2\pi f_D$ is the maximum Doppler shift in rad/s. (The remaining parameters have been already described in section II.) For an isotropic scattering (θ uniformly distributed between 0 and 2π), omnidirectional receiving antenna ($D\left(\theta\right) = 1$) and T following a negative exponential distribution, (4) reduces to [10]

$$\rho_2(\tau) = \frac{J_0^2(\omega_D\tau)}{1+(\Delta\omega\bar{T})^2} \quad (5)$$

where $J_0(\cdot)$ is the Bessel function of the first kind and zeroth order.

In (5), only frequency diversity is considered. Next, this expression shall be appropriately generalized to account for space diversity. As a rule, different instances of fading RPs are specified in terms of a time delay τ between them. Alternatively, instead of time, a distance separation may be used. Following this approach and knowing that a mobile at speed v moves a distance $d = v\tau$ in τ seconds (of course), the Rayleigh CCC (5) can be rewritten as

$$\rho_2(\tau = d/v) = \frac{J_0^2(2\pi d/\lambda)}{1+(\Delta\omega\bar{T})^2} \quad (6)$$

Again, note from (5) and (6) that d and τ are dual quantities which can be interchangeably used to specify instance separations of fading processes. Exploring this, the question now reduces to finding the equivalent distance d_e between antenna 1 at time instance t and antenna 2 at time instance $t+\tau$ and using $d = d_e$ in (6). From the geometry of Fig. (1), d_e can be easily found as

$$d_e = \sqrt{(v\tau)^2 + d_a^2 - 2v\tau d_a\cos\alpha} \quad (7)$$

Then, replacing d in (6) by (7), the CCC between Rayleigh processes at horizontally spaced omnidirectional antennas at the mobile station is given as in (2). Using (2) as ρ_2 into (3), (1) is finally attained.

From (1), the CCC $\rho(\tau)$ between $R_1(t)$ and $R_2(t)$ can be found as

$$\begin{aligned}\rho(\tau) &\triangleq \frac{E[R_1(t)R_2(t+\tau)] - E[R_1(t)]E[R_2(t+\tau)]}{\sqrt{\mathrm{Var}[R_1(t)]\mathrm{Var}[R_2(t+\tau)]}} \\ &= \frac{\Gamma(m_1+\frac{1}{2})\Gamma(m_2+\frac{1}{2})}{\sqrt{\Gamma(m_1)\,\Gamma(m_1+1) - \Gamma(m_1+\frac{1}{2})^2}} \\ &\quad \times \frac{\left({}_2F_1\left(-\frac{1}{2},-\frac{1}{2},m_2,\rho_2(\tau)\right)-1\right)}{\sqrt{\Gamma(m_2)\,\Gamma(m_2+1)-\Gamma(m_2+\frac{1}{2})^2}} \qquad (8)\end{aligned}$$

with $\rho_2(\tau)$ given in (2). ($\mathrm{Var}[\cdot]$ denotes variance.) With the aid of [1, Eq. (139)], an accurate, simple approximation to (8) can be found as

$$\rho(\tau) \approx \sqrt{\frac{m_1}{m_2}}\rho_2(\tau) \qquad (9)$$

Indeed, it can be shown that $\sqrt{m_1/m_2}\rho_2(\tau)$ is the exact correlation coefficient between $R_1^2(t)$ and $R_2^2(t)$ [8]. It is interesting to note that $\rho(\tau)$ is limited (approximately) to $\sqrt{m_1/m_2}$, since $0 \le \rho_2(\tau) \le 1$. In particular, for identical fading parameters ($m_1 = m_2$), $0 \le \rho(\tau) \le 1$, as expected.

IV. Channel Characterization

In the following, using (8) and (9), the coherence bandwidth and the coherence time/distance of the Nakagami-*m* channel are obtained.

A. Coherence Bandwidth

The coherence bandwidth B_c is defined as the frequency separation for which the envelope CCC equals a certain threshold ρ_{TH}. A well-accepted criterion is $\rho_{TH} = 0.5$. Solving (8) for $\Delta\omega$, with $\rho(\tau) = \rho_{TH}$ and $\tau = d_a = 0$, the coherence bandwidth in rad/s is found in a exact manner. This is a transcendental equation and must be solved numerically. On the other hand, using (9), an accurate approximation is obtained as

$$(B_c\bar{T})^2 \approx \begin{cases} \rho_{TH}^{-1}\sqrt{m_1/m_2} - 1 & , \quad \rho_{TH}^2 \le m_1/m_2 \\ 0 & , \quad \text{otherwise} \end{cases} \qquad (10)$$

Note that B_c increases with m_1/m_2. More interestingly, no bandwidth is required ($B_c = 0$) if $\rho_{TH}^2 > m_1/m_2$. In particular, for identical fading parameters ($m_1 = m_2$), (10) equals the coherence bandwidth of the Rayleigh channel [10].

B. Coherence Time/Distance

The coherence time T_c and the coherence distance D_c are defined as the time delay and the antenna spacing for which the envelope CCC equals zero, respectively. Solving (8) numerically for τ, with $\rho(\tau) = \Delta\omega = d_a = 0$, the coherence time is found in a exact manner. Using (9), an accurate approximation is obtained as $T_c \approx 2.40483/(2\pi f_D)$. (The first null of the Bessel function $J_0(\cdot)$ occurs at 2.40483.) In the same way, solving (8) numerically for d_a, with $\rho(\tau) = \Delta\omega = \tau = 0$, the coherence distance is obtained. Again, from (9), $D_c \approx 2.40483\lambda/(2\pi) \approx 0.383\lambda$. As expected, T_c and D_c are dual parameters satisfying $D_c = vT_c$. Our results show that the coherence time/distance of the Nakagami-*m* channel is practically identical to that of the Rayleigh channel [10].

V. Numerical Results and Examples

In this section, using (9), some illustrative CCC curves for Nakagami-*m* fading processes are obtained. In Fig. 2, the coherence bandwidth is plotted versus $\sqrt{m_1/m_2}$ for $\rho_{TH} = 0.1, 0.2, \ldots, 0.9$. B_c increases as m_1/m_2 increases and as ρ_{TH} decreases. Note from the curves that, for $m_1/m_2 \le \rho_{TH}^2$, B_c is nil.

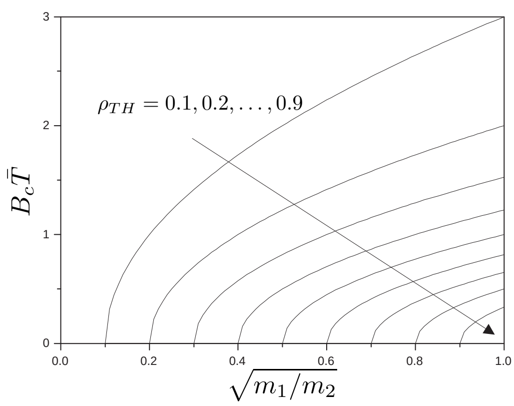

Fig. 2. Coherence Bandwidth.

Figs. 3 and 4 depict the influence of frequency separation and time delay on the crosscorrelation, respectively. Some plots of $\rho(0)$ against $\Delta\omega\bar{T}$ are shown in Fig. 3 for $\sqrt{m_1/m_2} = 0.1, 0.2, \ldots, 1$ and $d_a = 0$. As expected from (2) and (9), $\rho(0)$ decreases as $\Delta\omega\bar{T}$ increases and/or as m_1/m_2 decreases, being limited to $\sqrt{m_1/m_2}$. In Fig. 4, $\rho(\tau)$ is plotted for $\sqrt{m_1/m_2} = 0.1, 0.2, \ldots, 1$ and $d_a = \Delta\omega = 0$. Note that the nulls are independent of $\sqrt{m_1/m_2}$, with the first null corresponding to the coherence time.

The effects of antenna spacing/angle are considered in Figs. 5 and 6, for nil frequency separation. Fig. 5 shows CCC versus time and antenna spacing, with $\alpha = \pi/4$. Note that $\rho(\tau)$ is a commutative function of $f_D\tau$ and d_a/λ, which is in agreement with the time/space duality already mentioned in previous sections. In Fig. 6, with $d_a/\lambda = 1/2$, CCC is plotted against time and antenna angle. It can be observed that, on average, $\rho(\tau)$ increases as α decreases. In particular, for $\alpha = 0$, full correlation

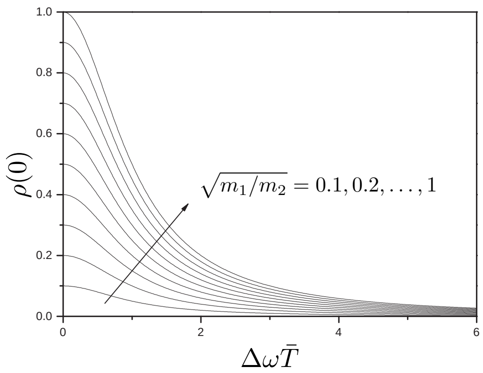

Fig. 3. Crosscorrelation coefficient versus frequency separation ($d_a = 0$).

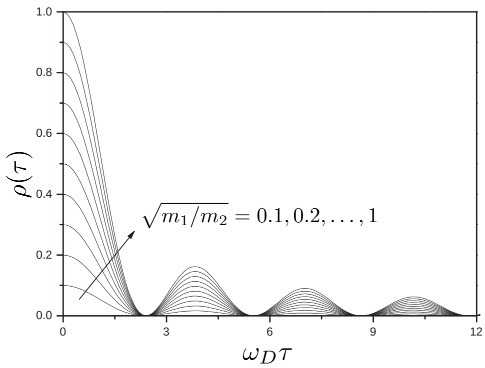

Fig. 4. Crosscorrelation coefficient versus time delay ($d_a = \Delta\omega = 0$).

occurs if $\tau = d_a/(f_D\lambda) = d_a/v$, which is the time interval the mobile station takes to move a distance equal to d_a.

VI. Conclusions

The generic-order CCF and the CCC between two non-identical Nakagami-*m* fading processes were derived, assuming a space-frequency diversity scenario with horizontally spaced omnidirectional antennas at the mobile station. In addition, we presented a simple, insightful, accurate approximation to the CCC. Based on it, the coherence bandwidth and the coherence time/distance of the Nakagami-*m* channel were also obtained. The coherence bandwidth increases with m_1/m_2 and is nil for $\rho_{TH}^2 > m_1/m_2$. On the other hand, the coherence time/distance is practically identical to that of the Rayleigh channel.

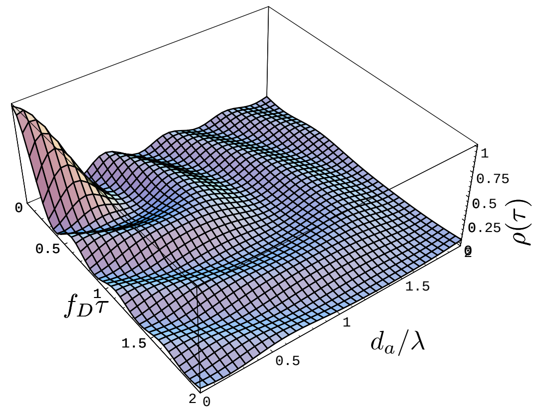

Fig. 5. Crosscorrelation coefficient versus time delay and antenna spacing ($\Delta\omega = 0$, $\alpha = \pi/4$).

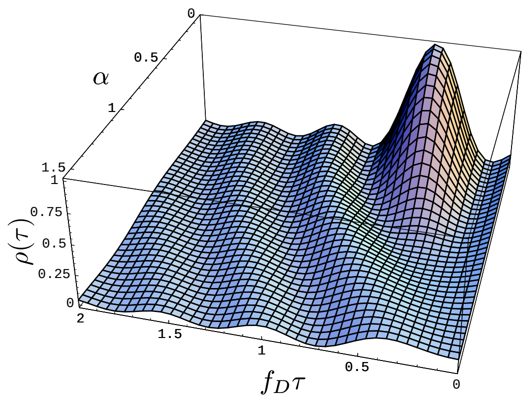

Fig. 6. Crosscorrelation coefficient versus time delay and antenna angle ($\Delta\omega = 0$, $d_a/\lambda = 1/2$).

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